Problem 3: Numerical methods for solving singular integral equations with Cauchy-type kernels

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Problem statement

Integral equations are equations in which some unknown function to be determined appears under one or several integral signs [1, 2]. The name integral equation was given by du Bois-Reymond in 1888. Integral equations arise in several fields of science; for example, in queuing theory , medicine, acoustics, heat and mass transfer, economics [3]. There are many types of integral equations. The classification of integral equations depends mainly on the limits of integration and the kernel of the equation. More details about integral equations and their origins can be found in [4, 5].

In this work, we will focus our concern on equations of the form:

$$\fint_{-1}^{1} \frac{\varphi(t)}{t-x}\,\mathrm{d}t + \int_{-1}^{1} k(x,t)\,\varphi(t)\,\mathrm{d}t = f(x)\ , \qquad -1 < x < 1\ , \tag{1}$$

where the kernel function $k(x,t)$ and the forcing function $f(x)$ are prescribed and the function $\varphi(x)$ is the unknown function to be determined. Equation (1) is called a Cauchy-type singular integral equation of the first kind and presents a Cauchy-type singularity at $t = x$. Singular internal equations with Cauchy kernels appear in many practical problems of elasticity, crack theory, wing theory and fluid flow [6]. The simplest singular integral equation of the first kind has the form

$$\fint_{-1}^{1} \frac{\varphi(t)}{t-s}\,\mathrm{d}t = f(x)\ , \qquad -1 < x < 1\ . \tag{2}$$

Equation(2) is called the characteristic singular integral equation and it is obtained when $k(x,t) = 0$ in (1). Equation (2) is also known as the airfoil equation in aerodynamics. The integral in (2), which is understood in the principal value sense, is defined as

$$\fint_{-1}^{1} \frac{\varphi(t)}{t-x}\,\mathrm{d}t = \lim_{\epsilon\to 0}\left[\int_{-1}^{x-\epsilon} \frac{\phi(t)}{t-x}\,\mathrm{d}t + \int_{x+\epsilon}^{1} \frac{\varphi(t)}{t-x}\,\mathrm{d}t\right]\ , \qquad -1 < x < 1\ . \tag{3}$$

The closed-form solution of the characteristic singular integral equation (2), which is unbounded at both end-points $x = \pm 1$, is given by the formula

$$\varphi(x) = -\frac{1}{\pi^2\sqrt{1-x^2}} \fint_{-1}^{1} \frac{\sqrt{1-t^2}\,f(t)}{t-x}\,\mathrm{d}t + \frac{C}{\pi\sqrt{1-x^2}}\ , \tag{4}$$

where

$$C = \int_{-1}^{1} \varphi(t)\,\mathrm{d}t\ . \tag{5}$$

Equation (2) can also be solved to obtain an approximate analytical solution or a numerical solution. Our aim in this project is to study numerical solutions of a simple integral differential equation of the form (2).

References

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